STURM'S THEOREM AND THE ZEROS OF A SOLUTION TO A DIFFERENTIAL EQUATION

WILLIAM A. OVERBAY

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ABSTRACT

The number of zeros contained in a given interval (a,b) for a solution to a Sturm-Liouville differential equation is of importance in many problems of mathematical physics. This number may be determined through Sturm's Comparison Theorem. Given one zero of the solution to a Sturm-Liouville differential equation, a technique, based upon Sturm's Theorem, of computing the next consecutive zero of the solution is proposed. The existence of a function which satisfies the desired end results of the proposed technique is shown. The technique is then applied to Bessel's differential equation and the results tabulated for the first 20 roots of $J_0(x)$ and $J_1(x)$. Unfortunately this technique did not achieve the desired result of convergence to successive zeros of the given Bessel Function.

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TABLE OF SYMBOLS

Symbol	Definition
y'	The first derivative of y with respect to the independent variable x.
y''	The second derivative.
$\|a-b\|$	The absolute value of the difference between a and b.
$\{x_i\}$	A sequence of elements x_i having some defined common characteristic.
(1.1)	Referring to equation numbered (1.1).
$a \equiv b$	a is identically equal to b.
g.l.b.	Greatest lower bound.
$a \doteq b$	a is approximately equal to b.
$d \ll c$	c is much greater than d.

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1. Introduction.

Consider a differential equation system.

$$[r(x)\, y'(x)]' + P(x)\, y(x) = 0 \qquad (1.1)$$

$$y(a) = 0 \qquad y(b) = 0$$

where $r(x)$ and $p(x)$ are positive, continuous, real valued functions on $[a,b]$. The system (1. 1) is known as a Sturm-Liouville System.

A frequent problem encountered in mathematical physics requires the determination of consecutive characteristic roots of a system (1. 1). The solutions to (1. 1) usually do not possess the nice regular characteristics of the trigonometric functions; consequently the former are in general more difficult to manipulate.

When the value of one characteristic root a of (1. 1) is known, Sturm's Comparison Theorem suggests a possible means of computing the next consecutive root b in the following sense:

A sequence of functions is generated from (1. 1). Each element of the sequence is obtained by selecting a particular value of the independent variable $x = x_i$ and substituting this value into the coefficient $P(x)$ and $r(x)$ of (1. 1) giving

$$[r(x_i)y'(x)]' + P(x_i)\, y(x) = 0 \qquad (1.2).$$

The solution of (1. 2) is forced to vanish at $y = a$, and the next consecutive zero of this solution is chosen as x_{i+1}, the next element of the sequence. It is desired that the sequence so generated result in an equation of the form (1. 2) such that its solution vanishes at $x = a$ and $x = b$, the zeros of the solution of (1. 1).

The theory is developed for a general Sturm-Liouville System but specific applications are limited to Bessel's Equation of order n. (n a positive integer).

We shall define an indexing of the roots of a solution to a differential equation as follows:

Let r_1 be the first root > 0, and r_2 the next root greater than r_1. Furthermore, if r_i is the i th root > 0, then r_{i+1} is the (i+1)st root > 0.

If r_k is the given root of a solution to a differential equation, we shall define r_{k+1} as the next consecutive root of the solution.

2. Mathematical Background.

This section contains theorems and mathematical developments which will be called upon in later sections.

2.1. Sturm's Comparison Theorem.

Given two equations of a Sturm-Liouville type:

$$y'' + A\,y = 0 \qquad (2.1.1)$$

$$y(a) = 0$$

$$u'' + Bu = 0 \qquad (2.1.2)$$

$$u(a) = 0$$

where A and B are positive, real constants and B is greater than A.

Let us determine a solution $y(x)$ to (2.1.1) such that $y(a) = 0$. The elementary theory of differential equations tells us that the general solution is of the form

$$y(x) = C_1 \sin\sqrt{A}x + C_2 \cos\sqrt{A}\,x \; .$$

When $x = a$

$$y(a) = C_1 \sin\sqrt{A}\,a + C_2 \cos\sqrt{A}\,a = 0$$

$$C_2 = -\frac{C_1 \sin\sqrt{A}\,a}{\cos\sqrt{A}\,a} \; .$$

Since C_1 is an arbitrary constant, choose $C_1 = k_o \cos\sqrt{A}\,a$.

$$y(x) = k_o\,[\sin\sqrt{A}\,x \cos\sqrt{A}\,a - \cos\sqrt{A}\,x \sin\sqrt{A}\,a\,]$$

$$y(x) = k_o \sin\sqrt{A}\,(x - a).$$

Similarly, (2.1.2) will have a solution $u(x) = k_1 \sin\sqrt{B}(x - a)$. The assumption $A < B$ tells us $u(x)$ will "oscillate" more rapidly than $y(x)$ or that $u(x)$ will have at least as many zeros as $y(x)$ on a given interval (a, b).

When the coefficients of (2.1.1) are positive, real valued functions of the independent variable x, the solutions obtained must be defined in terms of so-called "special" functions such as Bessel functions.

Charles Sturm (1803-1855) made a study of the rates of "oscillation" of solutions to equations of the form

$$[\ r\ (x)\ y'(x)\ ]' + P\ (x)\ y(x) = 0$$

$$[\ r\ (x)\ w'(x)\ ]' + Q(x)\ w(x) = 0$$

$$y(a) = w(a) = y(b) = 0$$

where $r(x) > 0$; $P(x)$ and $Q(x)$ are continuous on $[a,b]$. He proved that, in general, the larger $P(x)$ the faster $y(x)$ will oscillate. More concisely, Sturm's Comparison Theorem is stated as follows:

Theorem 2.1 Sturm's Comparison Theorem.

Given two equations:

$$[r(x)\ y']' + p(x)\ y = 0 \tag{2.1.3}$$

and $$[r(x)\ v']' + q(x)\ v = 0 \tag{2.1.4}$$

where $r(x)$ is positive in the closed interval $[a,b]$; $r(x)$, $p(x)$ and $q(x)$ are continuous in $[a,b]$ and $q(x) \geq p(x)$ with strict inequality for at least one point in $[a,b]$. If a solution $y(x)$ of (2.1.3) has consecutive zeros at $x = x_o$ and $x = x_1$, with $x_1 > x_o$, and x_1, x_o are in the interval $[a,b]$, then a solution $v(x)$ of (2.1.4), which vanishes at $x = x_o$, will vanish again in the open interval (x_o, x_1).

Proof:

Suppose $y(x) > 0$ in (x_o, x_1) and $y'(x_o) > 0$, $y'(x_1) < 0$ and $v'(x_o) > 0$. Then if $y(x)$ and $v(x)$ are solutions to (2.1.3) and (2.1.4) respectively, we have the identities

$$[r(x)\ y'(x)]' + p(x)\ y(x) = 0 \tag{2.1.5}$$

and $$[r(x)\ v'(x)]' + q(x)\ v(x) = 0 \tag{2.1.6}$$

If we multiply equation (2.1.5) by $-v(x)$ and equation (2.1.6) by $y(x)$ we have

$$-[r(x)\, y'(x)]'\, v(x) - p(x)\, y(x)\, v(x) = 0 \qquad (2.1.7)$$

$$[r(x)\, v'(x)]'\, y(x) + q(x)\, v(x)\, y(x) = 0 \qquad (2.1.8)$$

Adding (2.1.7) and (2.1.8) then integrating from x_o to x_1 we have

$$r(x)[y(x)v'(x) - y'(x)v(x)] \Big|_{x_o}^{x_1} + \int_{x_o}^{x_1} [q(x)-p(x)]y(x)v(x)dx = 0$$

or

$$r(x_1)y'(x_1)v(x_1) = \int_{x_o}^{x_1} [q(x)-p(x)]\, y(x)\, v(x)\, dx \qquad (2.1.9)$$

since $y(x_o) = 0$ if x_o is a solution to (2.1.3).

Suppose now that $v(x) > 0$ everywhere in (x_o, x_1), (i.e. if $v(x_o) = 0$ and $v'(x_o) > 0$ then $v(x)$ has not vanished anywhere in (x_o, x_1)), then this implies that $r(x_1)\, v(x_1)\, y'(x_1)$ is negative while $\int_{x_o}^{x} (q(x) - p(x))\, y(x)\, v(x)\, dx$ is positive. This contradiction implies that $v(x)$ cannot be positive everywhere in (x_o, x_1). Similarly, the assumption $v'(x_o) < 0$ and $v(x)$ negative everywhere on (a, b) again leads to a contradiction and the theorem is established.

2.2 The Normal Form.

Consider a differential equation

$$y'' + P(x)\, y' + Q(x)\, y = 0 \qquad (2.2.1).$$

Let $y = u\, v$, then $y' = uv' + u'v$

and

$$y'' = uv'' + 2u'v' + u''v.$$

Substituting for y in (2.2.1) we have

$$uv'' + [2u' + P(x)u]v' + [u'' + P(x)\, u' + Q(x)u]\, v = 0 \qquad (2.2.2).$$

Suppose we choose u such that the coefficient of v' vanishes. Set

$$2u' + P(x)\, u = 0$$

$$du/u + \frac{P(x)}{2}\, dx = 0$$

$$u = C \exp\left[-\frac{1}{2} \quad \frac{P(x)}{2} \quad dx\right]$$

Let C = 1 and we have

$$u' = \frac{1}{2} P(x)\, u$$

$$u'' = \frac{1}{2} P(x)\, u' - \frac{1}{2} u\, {}^{dP}/_{dx}$$

$$= + \frac{(P(x))^2}{4} u - \frac{1}{2}\, {}^{dP}/_{dx}\, u\,.$$

Substituting u' and u'' in (2. 2. 2) we get

$$uv'' + [-P(x)\, u + P(x)\, u]\, v' + [Q(x)u + \frac{(P(x)}{4}^2 u - \frac{1}{2}\, {}^{dP}/_{dx}\, u$$

$$- \frac{P(x)^2}{2} u\,]\, v = 0\,.$$

We see that we have a factor of u in every term,therefore it may be divided out leaving

$$v'' + [Q(x) - \frac{P(x)^2}{4} - \frac{1}{2} \frac{dP(x)}{dx}\,]\, v = 0 \qquad (2.2.3)$$

We say that (2. 2. 3) is equation (2. 2. 1) transformed into normal form. We will see that the normal form plays an important role in our later development.

2. 3. Existence Theorem.

Consider a differential equation reduced to normal form

$$y'' + R(t)\, y = 0. \qquad (2.3.1)$$

$$y(a) = y(b) = 0.$$

We desire to know if there exists a $\bar{t}$ in the interval (a,b) such that the equation

$$u'' + R(\bar{t})\, y = 0 \qquad (2.3.2)$$

$R(\bar{t})$ is $R(t)$ evaluated at $t = \bar{t}$ such that a solution $u(t)$ to (2. 3. 2) can be forced to have zeros at both $t = a$ and $t = b$. The answer to this question may be stated in the form of a theorem as follows:

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$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

Let $R(Y)$ [illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

We can then [illegible]

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$$[illegible] \qquad [illegible]$$

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[illegible]

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[illegible]

Consider [illegible]

$$[illegible] \qquad [illegible]$$

$$[illegible]$$

[illegible]

[illegible]

$$[illegible] \qquad [illegible]$$

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Theorem 2.2.

Given a differential equation of the form:

$$y'' + R(t)\, y = 0 \qquad (2.3.3)$$

where $R(t)$ is positive, real valued, continuous and monotone in (a, c), where $b \leq c$, and whose solution $y(t)$ has consecutive zeros at $t = a$ and $t = b$, $a < b < c$. Then there exists a $\bar{t}$ in the interval (a,b) such that the equation

$$u'' + R(\bar{t})\, u = 0 \qquad (2.3.4)$$

has a solution $u(t)$ with a zero at $t = a$ and moreover, $u(t)$ has its next consecutive zero at $t = b$.

Proof:

First assume $R(t)$ is monotone decreasing, then $R(a) \geq R(t) \geq R(b)$ for all t in (a,b). Choose that solution $u_b(t)$ of $u'' + R(b)\, u = 0$ such that $u_b(t)$ vanishes at $t = a$. A solution that satisfies this requirement is $u_b(t) = k_b \sin \sqrt{R(b)}\,(t-a)$. Clearly $u_b(a) = 0$ and moreover, $u_b(t)$ will have its next consecutive zero, for t greater than a, when the argument of the sine function is equal π, that is, when $\sqrt{R(b)}\,(t-a) = \pi$. Since R is assumed decreasing, $R(b) \leq R(t)$ for all t in (a,b) therefore, Sturm's Comparison Theorem tells us that $u_b(x)$ will vanish again at $t = t_1$ where $b \leq t_1$. Similarly, let $u'' + R(a)\, u = 0$ have a solution $u_a(x)$ which vanishes for $t = a$. Then with $R(t) \leq R(a)$ for all t in the interval (a,b), we have that $u_a(t)$ will vanish again at some point $t = t_2$ such that $a < t_2 < b$.

Case II.

Now suppose $R(t)$ is increasing. Choose $u_b(t)$ a solution of

$$v'' + R(b)\, v = 0$$

such that a root at $t = a$. $R(t) \leq R(b)$ for all t in (a,b), hence $v_b(t)$

will vanish again when $t = t_3$, $t_3 < b$. Similarly, $v'' + R(a)\, v = 0$ has a solution $v_a(t)$ which vanishes at $t = a$ and will have its next consecutive zero at some $t = t_4$, $b < t_4$.

In either case, $R(t)$ increasing or $R(t)$ decreasing, the root b of equation (2.3.3) is between two roots of (2.3.4) determined by evaluating $R(t)$ at $t = a$ and at $t = b$. This and the continuity of $R(t)$ assures us that there exists a $\bar{t}$ in (a, b) such that $u_{\bar{t}}(t)$, a solution to (2.3.4) which vanishes at $t = a$, will vanish again when $t = b$, and the theorem is established.

An interesting conclusion from Theorem 2.2 is the following:

Theorem 2.2.1.

Given a system of the type (2.3.1) where $R(t)$ is monotone, then the next consecutive root of a solution $u(t)$ to (2.3.4) such that $u(a) = 0$ must lie in the interval $\left(a + \dfrac{\pi}{\sqrt{R(b)}}\ ,\ a = \dfrac{\pi}{\sqrt{R(a)}}\right)$.

The proof follows directly from the proof of Theorem 2.2. The end points of the interval $(a + \pi/\sqrt{R(b)},\ a + \pi/\sqrt{R(a)})$ are precisely the t_i found in Theorem 2.2.

Note that the arbitrary constants occuring in the solutions to equations (2.3.1) and (2.3.4) have no effect upon the zeros and, therefore, will be ignored in all further considerations.

We should also note that in a general Sturm-Liouvile System (2.3.1), the function $R(t)$ is not always monotone.

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3. Bessel's Equation.

Let us now consider a typical Sturm - Liouville System:

$$x^2 y'' + xy' + (x^2 - n^2)\, y = 0 \tag{3.1}$$

$$y(a) = y(b) = 0 \tag{3.1a}$$

Equation (3.1) is the general form of Bessel's Differential Equation of order n. A solution of (3.1) is

$$y(x) = J_n(x) \tag{3.2}$$

where $J_n(x)$ is known as the Bessel function of the first kind.

We see that before we can derive any benefit from the theorems of section 2, we must reduce (3.1) to its normal form. Write (3.1) in the form

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0. \tag{3.3}$$

Section 2.2 tells us that (3.3) may be reduced to normal form by the transformation

$$\begin{aligned} y &= v \exp\left[-\frac{1}{2} \int \frac{1}{x}\, dx \right] \\ &= v \exp\left[-\frac{1}{2} \log_e x \right] \\ &= v\, x^{-1/2} \end{aligned} \tag{3.4}$$

Then

$$y' = -\frac{1}{2} v(x)^{-3/2} + v'\, x^{-1/2} \tag{3.5}$$

$$y'' = \frac{3}{4} v\, x^{-5/2} - \frac{1}{2} v' x^{-3/2} - \frac{1}{2} v' x^{-3/2} + v'' x^{-1/2} \tag{3.6}$$

Substituting (3.4), (3.5), and (3.6) into (3.3) and simplifying we have the desired normal form of Bessel's Equation

$$v'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right) v = 0. \tag{3.7}$$

The transformation (3.4) implies that (3.7) has a solution

$$v(x) = x^{1/2} J_n(x). \tag{3.8}$$

Trivially any zero of (3.2) is also a zero of (3.8).

Consider the differential equation

$$w'' + w = 0 \tag{3.9}$$

$$w(a) = 0$$

We know that a solution to (3.9)

$$w(x) = k \sin(x-a)$$

vanishes at $x = a$ and again at $x = a + \pi$.

Suppose we examine (3.7) and (3.9) with respect to Theorem 2.1 (Sturm's Comparison Theorem). Let us denote the coefficient of v in (3.7) as

$$P(x) = 1 + \frac{1-4n^2}{4x^2} \tag{3.10}$$

We see from (3.10) that we must consider two cases, 1) when $n = 0$ and 2) when $n \geq 1$.

Case I.

$n = 0$. We see that $P(x) \geq 1$ and continuous, therefore, theorem 2.1 applies. We conclude that if the solution (3.8) to (3.7) vanishes when $x = a$, it will vanish again on the interval $(a, a + \pi)$. This implies that the difference between consecutive roots of $J_o(x)$ is less than π. Moreover, we see that

$$\lim_{x \to \infty} P(x) = 1 .$$

This tells us that with respect to the index i of roots r_i ;

$$\lim_{o \to \infty} \left| r_{i+1} - r_i \right| = \pi . \tag{3.11}$$

Case II.

$n \geq 1$. In this case $0 < P(x) < 1$ when $\frac{1}{2}\sqrt{4n^2 - 1} < x < \infty$. Then x must be in the interval $(\frac{1}{2}\sqrt{4n^2-1}, \infty)$ in order that (3.7) have an oscillating solution. Theorem 2.1 tells us that a solution to (3.7) which vanishes at $x = a$ will not vanish again in $(a, a + \pi)$.

Therefore, the difference between the consecutive roots of $J_n(x)$ is greater than π. Similarly,

$$\lim_{x \to \infty} P(x) = 1, \text{ when } n \geq 1,$$

tells us that the limiting difference is the same as in Case I, namely the difference given in (3.11).

The treatment of the differences between consecutive roots of the Bessel functions in the above two cases, suggests that further treatment of the problem also be made in two parts, i.e. 1) $n = 0$ and 2) $n \geq 1$.

4. Application to Bessel's Equation of Order Zero.

When $n = 0$, the normal form of Bessel's differential equation reduces to

$$v'' + \left(1 + \frac{1}{4x^2}\right) v = 0. \tag{4.1}$$

Suppose the conditions

$$v(a) = v(b) = 0 \tag{4.2}$$

are imposed. Let the coefficient of v in (4.1) be denoted by $P(x)$, i.e.

$$P(x) = 1 + \frac{1}{4x^2} \tag{4.3}$$

Clearly $P(x) > 1$ for all x, $P(x)$ is continuous and decreasing. Then, from theorem 2.2, there exists an $\bar{x}$ in the interval (a,b) such that the equation

$$u'' + \left(1 + \frac{1}{4\bar{x}^2}\right) u = 0$$

has a solution

$$u(x) = k \sin \sqrt{1 + \frac{1}{4\bar{x}^2}}\ (x-a)$$

with a root at $x = a$ and, moreover, will have the next consecutive root at $x = b$.

Let us now attempt to construct a sequence $\{x_i\}$ which will converge to $\bar{x}$. There are two immediate choices for the first element of the sequence, a or $a + \pi$. We will generate the sequence $\{x_i\}$ by means of the following recursion formulae:

$$P(x_i) = 1 + \frac{1}{4x_i^2} \tag{4.4}$$

$$u_i'' + P(x_i)\, u_i = 0 \tag{4.5}$$

$$u_i(x) = k_i \sin \sqrt{P(x_i)}(x-a) \tag{4.6}$$

$$x_{i+1} = a + \pi / \sqrt{P(x_i)} \tag{4.7}$$

Wherein (4.6) is a solution to (4.5) which is forced to have a root at $x = a$; and x_{i+1} is the next consecutive root of (4.6). We see from (4.4) that $P(x_i) > 1$ for all x_i. This implies that $x_{i+1} < a + \pi$ for all x_i.

Case I.

Choose $x_o = a$, giving

$$P(x_o) = 1 + \frac{1}{4x_o^2} = 1 + \frac{1}{4a^2} \tag{4.8}$$

$$u_o'' + P(x_o)\, u_o = 0 \tag{4.9}$$

$$u_o(x) = k_o \sin\sqrt{P(x_o)}(x-a) \tag{4.10}$$

$$x_1 = a + \pi / \sqrt{P(x_o)} \tag{4.11}$$

Since $P(a) \geq P(x)$ for all x in the interval (a,b), theorem (2.1) (Sturm's Comparison Theorem) tells us that x_1 lies in the interval $a < x_1 < b$.

Then x_1 is substituted in the recursion equations giving

$$P(x_1) = 1 + \frac{1}{4x_1^2} \tag{4.12}$$

$$u_1'' + P(x_1)\, u_1 = 0 \tag{4.13}$$

$$u_1(x) = k_1 \sin \sqrt{P(x_1)}\,(x-a) \tag{4.14}$$

$$x_2 = a + \pi / \sqrt{P(x_1)} \tag{4.15}$$

We cannot apply theorem 2.1 with respect to (4.13) and (4.3) since we cannot guarantee the necessary inequality between $P(x_1)$ and $P(x)$. However, we may apply theorem 2.1 with respect to (4.13) and (4.9) since $P(x_1) < P(x_o)$. This tells us that $x_2 > x_1$.

If we expand (4.15)

$$x_2 = a + \frac{\pi}{\sqrt{1 + \frac{1}{4x_1^2}}} = a + \frac{\pi}{\sqrt{1 + \frac{1}{4\left(a + \frac{\pi}{\sqrt{1 + \frac{1}{4a^2}}}\right)^2}}}$$

we see the sequence is generating a rather strange appearing continued fraction; strange in the sense that the more common continued fractions do not contain radical expressions. Moreover, note that the fractional form becomes exceedingly more complicated as each new term is added.

Let us now assume that for some $m > 0$, we have

$$x_{m-1} > x_{m-2} \tag{4.16}$$

The inequality (4.16) implies that the following inequality also holds:

$$P(x_{m-1}) = 1 + \frac{1}{4x_{m-1}^2} < P(x_{m-2})\ 1 + \frac{1}{4\,x_{m-2}^2} \tag{4.17}$$

Theorem 2.1 immediately tells us that x_m is greater than x_{m-1}. Inductively we have shown that the sequence $\{x_i\}$ is increasing. Moreover, we have shown that the sequence is bounded above by $a + \pi$, consequently, the sequence converges to some $\tilde{x}$ such that

$$\tilde{x} = a + \frac{\pi}{\sqrt{1 + \frac{1}{4\tilde{x}^2}}}$$

$$(\tilde{x}-a)^2 = \frac{4\tilde{x}^2\,\pi^2}{4\tilde{x}^2 + 1}$$

$$(4\tilde{x}^2 + 1)(\tilde{x} - a)^2 - 4\tilde{x}^2\,\pi^2 = 0 \tag{4.23}$$

$$4\tilde{x}^4 - 8a\,\tilde{x}^3 + (4a^2 + 1 - 4\pi^2)\,\tilde{x}^2 - 2a\,\tilde{x} + a^2 = 0\,. \tag{4.24}$$

Then $\tilde{x}$ is the root of (4.24) that is nearest $a + \pi$. We see from (4.23) that for large roots, $4\,\tilde{x}^2 \gg 1$ and the large roots of (4.24) may be approximated by:

$$4\tilde{x}^2(\tilde{x}-a)^2 - 4\tilde{x}^2\pi^2 = 0$$

$$(\tilde{x}-a)^2 - \pi^2 = 0$$

$$\tilde{x} \doteq a \pm \pi$$

A simple calculation will show for any given root 'a' of $J_o(x)$ that (4.24) has a root near $a + \pi$.

Case II.

Let us choose $x_o = a + \pi$.

$$P(x_o) = 1 + \frac{1}{4(a+\pi)^2} \tag{4.25}$$

$$u_o'' + P(x_o)\, u_o = 0 \tag{4.26}$$

$$u_o(x) = k_o \sin \sqrt{P(x_o)}\,(x-a) \tag{4.27}$$

$$x_1 = a + \pi / \sqrt{P(x_o)} \tag{4.28}$$

We have $P(a+\pi) < P(x)$ for all x in (a,b), therefore, we know x_1 cannot be in (a,b), therefore, we see

$$b < x_1 < a + \pi \tag{4.29}$$

Substituting x_1 in the recursion formulae we have

$$P(x_1) = 1 + \frac{1}{4x_1^2} \tag{4.30}$$

$$u_1'' + P(x_1)\, u_1 = 0 \tag{4.31}$$

$$u_1(x) = k_1 \sin \sqrt{P(x_1)}\,(x-a) \tag{4.32}$$

$$x_2 = a + \pi / \sqrt{P(x_1)} \tag{4.33}$$

The inequality (4.29) tells us that $P(x_1) > P(x_o)$, and we get $x_2 < x_1$ from Theorem 2.1. Moreover, (4.29) also tells us that $P(x_1) < P(x)$ for all x in (a,b) and $x_2 > b$.

If we reverse the assumed inequality in (4.16) we can show inductively that this sequence $\{x_i\}$, generated by starting at $x_o = a + \pi$, is a monotone decreasing sequence. Moreover, we see that $\{x_i\} > b$

for all i, hence is bounded below. Then the sequence converges to an $\tilde{x}_1$ where

$$\tilde{x}_1 = a + \pi / \sqrt{P(\tilde{x}_1)}$$

$$= a + \frac{\pi}{\sqrt{1 + \frac{1}{4\tilde{x}_1^2}}}$$

$$(4\tilde{x}_1^2 + 1)(\tilde{x}_1 - a)^2 - 4\tilde{x}_1^2\pi^2 = 0$$

$$4\tilde{x}_1^4 - 8a\tilde{x}_1^3 + (4a^2 + 1 - 4\pi^2)\tilde{x}_1^2 - 2a\tilde{x}_1 + a^2 = 0 \qquad (4.34).$$

The quartic (4.34) has coefficients which are identical to the coefficients of like powers of $\tilde{x}$ in (4.24), therefore we conclude that $\tilde{x}_1 \equiv \tilde{x}$. This shows that there is no difference in the limit between the sequence generated in Case I and that generated in Case II. Unfortunately, we have also shown that $\tilde{x} > b$, but we have no indication of the difference between $\tilde{x}$ and b.

The quartic (4.24) has been solved with the aid of the CDC 1604 Digital Computer for the first 20 roots of $J_0(x)$. The computed values of $\tilde{x}$ are compared with the tabulated values of the roots of $J_0(x)$. The difference between $\tilde{x}$ and the tabulated value of b is shown in the error column. The difference has its maximum at the first root and decreases quite rapidly as x grows larger.

Simple calculations show that $\tilde{x}$ is almost midway between b and $a + \pi$ for the first root, but this relation does not persist as the root number increases. In fact $\tilde{x}$ approaches b more rapidly than $a + \pi$.

The fact that the sequence x_i does not converge to b implies that some change to the recursion formulae must be made. The most direct method of changing the formulae consists of a modification to $P(x)$.

4.1. Linear Average.

Suppose a sequence $\{x_i\}$ were to converge such that (4.1.1) holds, i.e.

$$b = a + \pi \Big/ \sqrt{1 + \frac{1}{4\mathcal{X}^2}} \qquad (4.1.1)$$

where $\mathcal{X}$ is the limit point of a sequence $\{x_i\}$.

$$(4\mathcal{X}^2 + 1)(b-a)^2 = 4\mathcal{X}^2 \pi^2$$

$$4\mathcal{X}^2[(b-a)^2 - \pi^2] = -(b-a)^2$$

$$\mathcal{X} = \frac{b-a}{2} \sqrt{\frac{-1}{(b-a)^2 - \pi^2}}$$

$$\mathcal{X} = \frac{b-a}{2\sqrt{\pi^2 - (b-a)^2}}.$$

We have shown that $(b-a) < \pi$, therefore, we know that $\mathcal{X}$ is real. Computed values of $\mathcal{X}$ are displayed in table II, along with the tabulated values of b. The error column is the difference between $\tilde{x}$ and $\mathcal{X}$, and indicates the amount $\tilde{x}$ must be decreased if the sequence $\{x_i\}$ is to converge to b. Noting that $\mathcal{X}$ is almost midway between a and $\tilde{x}$ the following modification to the recursion formulae is suggested:

$$P(\bar{x}_i) = 1 + \frac{1}{4\left(\frac{a + \bar{x}_i}{2}\right)^2} \qquad (4.1.2)$$

$$u_i + P(\bar{x}_i)\, u_i = 0 \qquad (4.1.3)$$

$$u_i(x) = k_i \sin \sqrt{P(\bar{x}_i)}\,(x-a) \qquad (4.1.4)$$

$$\bar{x}_{i+1} = a + \pi \Big/ \sqrt{P(\bar{x}_i)} \qquad (4.1.5)$$

In essence, the modification consists of merely substituting $\frac{(a + \bar{x}_i)^2}{2}$ in place of x_i in (4.4). Clearly the conclusions reached regarding convergence of the sequence $\{x_i\}$ are also valid for the

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sequence $\{\bar{x}_i\}$. For the sequence $\{\bar{x}_i\}$ we are unable to make any conclusion relative to b since the $\bar{x}_1$ of Case II is

$$\bar{x}_1 = a + \pi / \sqrt{1 + \frac{1}{4\bar{x}_o^{\,2}}}$$

$$= a + \frac{\pi}{\sqrt{1 + \frac{1}{4(\frac{a+a+\pi}{2})^2}}}$$

$$= a + \frac{\pi}{\sqrt{1 + \frac{1}{4(a+\frac{\pi}{2})^2}}} \qquad (4.1.6)$$

and lies in the interval (a, b). Moreover, we can see that for $i > 1$ each element of the sequence $\{\bar{x}_i\}$ is smaller than the corresponding element of the sequence $\{x_i\}$ and we conclude that the limit of the sequence $\{\bar{x}_i\} = \bar{x}$ is smaller than $\tilde{x}$. We have that

$$\bar{x} = a + \frac{\pi}{\sqrt{1 + \frac{1}{4(\frac{a+\bar{x}}{2})^2}}} = a + \frac{\pi}{\sqrt{1 + \frac{1}{(a+\bar{x})^2}}}$$

$$(\bar{x} - a)^2[(a+\bar{x})^2 + 1] - (a+\bar{x})^2\,\pi^2 = 0 \qquad (4.1.7)$$

$$\bar{x}^4 + (1 - 2a^2 - \pi^2)\,\bar{x}^2 - [2a(1+\pi^2)]\,\bar{x} + a^4 + a^2(1-\pi^2) = 0 \qquad (4.1.8)$$

For the larger roots of (4.1.8) we look at (4.1.7) and see that $(a+\bar{x})^2 \gg 1$ so that an approximation yields

$$(\bar{x} - a)^2\ (a+\bar{x})^2 - (a+\bar{x})^2\,\pi^2 = 0$$

$$\bar{x} = a \pm \pi\,.$$

We conclude that (4.1.8) has a solution near $a + \pi$, and that this solution is the $\bar{x}$ we desire.

Table III lists the desired roots of (4.1.8) for the same 20 roots as in section 4. A comparison of the error column of table III against

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table I shows that the difference between $\bar{x}$ and the tabulated root b is about one-tenth as large as the difference between $\tilde{x}$ and b at the first root. Moreover, we see that $\bar{x}$ approaches b more rapidly than $\tilde{x}$, that $\bar{x} - b$ is only .01 times $\tilde{x} - b$ for the tenth root.

Based upon the results of these computations, we conclude that although $\bar{x}$ is nearer b than $\tilde{x}$, $\bar{x}$ is still not the value of x whose existence is guaranteed by theorem 2.2.

4.2. Mean Value.

Consider the integrated mean value of P(x) over the interval (a, x) and denote this integrated mean value by $\mathbb{P}(x)$, then

$$\mathbb{P}(x) = \frac{1}{x-a} \int_a^x \left(1 + \frac{1}{4t^2}\right) dt$$

$$= \frac{1}{x-a} \left[t - \frac{1}{4t}\right]_a^x$$

$$= \frac{1}{x-a} \left[x-a + \frac{1}{4}\left(\frac{1}{x} - \frac{1}{a}\right)\right]$$

$$= 1 + \frac{1}{4(x-a)} \left(-\frac{a-x}{ax}\right)$$

$$= 1 + \frac{1}{4ax}$$

We see that

$$\mathbb{P}(x) = 1 + \frac{1}{4ax} > P(\bar{x}) = 1 + \frac{1}{(a+x)^2} \tag{4.2.1}$$

$$\frac{1}{4ax} > \frac{1}{(a+x)^2}$$

$$a^2 + 2ax + x^2 > 4ax$$

$$a^2 - 2ax + x^2 > 0$$

$$(a-x)^2 > 0 \qquad \text{for all } x \neq a .$$

Suppose we generate a sequence $\{x_i\}$ by means of the following recursion formulae:

$$\mathbb{P}(x_i) = 1 + \frac{1}{4ax_i} \qquad (4.2.2)$$

$$u_i'' + \mathbb{P}(x_i)\, u_i = 0 \qquad (4.2.3)$$

$$u_i(x) = k_i \sin \sqrt{\mathbb{P}(x_i)}(x-a) \qquad (4.2.4)$$

$$x_{i+1} = a + \pi / \sqrt{\mathbb{P}(x_i)} \qquad (4.2.5)$$

But equations (4.2.2) through (4.2.5) are the same as the set of equations (4.4) through (4.7) except ax_i is substituted for x_i^2 in (4.4). If we make this substitution throughout our preceeding consideration we see that the sequence $\{x_i\}$ also converges to some $x = \hat{x}$ such that

$$\hat{x} = a + \frac{\pi}{\sqrt{1 + \frac{1}{4ax}}}$$

$$(4a\hat{x} + 1)(\hat{x}-a)^2 - 4a\hat{x}\,\pi^2 = 0$$

$$4a\hat{x}^3 + (1-8a^2)\,\hat{x}^2 + (4a^3 - 2a - 4a\,\pi^2)\,\hat{x} + a^2 = 0 \qquad (4.2.9)$$

Inequality (4.2.1) shows us that x must indeed be less than $\bar{x}$ but once more we are unable to state any analytical conclusions regarding x and b.

Table IV contains the roots of (4.2.9) for the same zeros of $J_0(x)$ as previously computed. We see that $\hat{x}$ is smaller than b. We note also that $|\hat{x}-b|$ is greater than $|\bar{x}-b|$.

Since b has been bracketed by $\tilde{x}$ and $\hat{x}$, a linear average of the two is suggested. Accordingly we define

$$P(x1_i) = (P(x_i) + P(\hat{x}_i))/2 \qquad (4.2.10)$$

$$x1_{i+1} = a + \pi / \sqrt{P(x1_i)} \qquad (4.2.11)$$

Then equation (4.2.10) and (4.2.11) together with equations (4.5) and (4.6) constitute a set of recursion formulae which will generate a

sequence $\{x1_i\}$. We know the sequence $\{x1_i\}$ will converge to some $\overline{x1}$ since it is the sum of two convergent sequences $\{x_i\}$ and $\{\hat{x}_i\}$. The values of $\overline{x1}$ are displayed in Table V. We see that although the values of $\overline{x1}$ are nearer to b than $\tilde{x}$, they are not as near as $\bar{x}$.

Suppose we consider the average of $\bar{x}$ and $\hat{x}$. Let us define:

$$P(x2_i) = (P(\bar{x}_i) + P(\hat{x}_i))/2 \qquad (4.2.12)$$

$$x2_{i+1} = a + \pi / \sqrt{P(x2_i)} \qquad (4.2.13)$$

Again equations (4.2.12) and (4.2.13) together with (4.5) and (4.6) will generate the convergent sequence $\{x2_i\}$. We see from Table VI that $\overline{x2}$ is nearer to b than any of the previous values generated. The maximum difference between $\overline{x2}$ and b is less than 10^{-3} for the first root and decreases to less than 10^{-7} for the 20th root. The difference decreases quite rapidly as is evidenced by a value less than 10^{-6} beyond the fifth root.

4.3 Cumulative Effect.

Let us now investigate the cumulative error for each of the sequences previously discussed. The errors in this case are cumulative in the following sense:

Assume that $a = r_1$, the first root of $J_o(x)$ is known, we will compute an x from a sequence. Then each successive root is generated using the root last computed as the value of the known root.

The cumulative errors for $\tilde{x}$, $\bar{x}$, $\hat{x}$, $\overline{x1}$ and $\overline{x2}$ are shown in Tables VII through XI respectively. As well expected, the cumulative error is due mostly to the error in computing the first root. We see from Table XI that the cumulative error for the 20th root is only 10%

greater than the error for the first root. The worst case is shown in Table X where the cummulative error for the 20th root is 40% greater than the error for the first root.

5. Application to Bessel's Equation of Order $n \geq 1$.

When $n \geq 1$, the normal form of Bessel's equation is

$$y'' + \left(1 + \frac{1-4n^2}{4x^2}\right) y = 0 \tag{5.1}$$

Let us impose the condition

$$y(a) = y(b) = 0 .$$

Again let $P(x)$ denote the coefficient of y in (5.1), i.e.

$$P(x) = 1 + \frac{1-4n^2}{4x^2} \tag{5.2}$$

We see that $P(x) > 0$ for all $x > \frac{1}{2}\sqrt{4n^2-1}$, moreover, $P(x)$ is monotone increasing, $\lim_{x \to \infty} P(x) = 1$. i.e. $0 < P(x) \leq 1$ for all x such that x is in the interval $\frac{1}{2}\sqrt{4n^2-1}, \infty)$. We shall limit our discussion to the case $P(x) > 0$ in order that the conditions of the theorems of section 2 may be satisfied.

Except for $P(x_i)$, the recursion formulae defining the sequence will be the same as in section 4, i.e.

$$P(x_i) = 1 + \frac{1-4n^2}{4x_o^2} \tag{5.3}$$

$$u_i'' + P(x_i)\, u_i = 0 \tag{5.4}$$

$$u_i(x) = k_i \sin\sqrt{P(x_i)}\,(x-a) \tag{5.5}$$

$$x_{i+1} = a + \pi/\sqrt{P(x_i)} \tag{5.6}$$

As before we consider the two cases $x_o = a$ and $x_o = a + \pi$.

Case I:

Choose $x_o = a$, giving

$$P(x_o) = 1 + \frac{1-4n^2}{4x_o^2} = 1 + \frac{1-4n^2}{4a^2} \tag{5.7}$$

$$u_o'' + P(x_o)\, u_o = 0 \tag{5.8}$$

$$u_o(x) = k_o \sin\sqrt{P(x_o)}\,(x-a) \tag{5.9}$$

$$x_1 = a + \pi / \sqrt{P(x_o)} \tag{5.10}$$

Since $P(x)$ is increasing, we see that $P(x_o) \leq P(x)$ for all x in (a,b). Then theorem 2.1 tells us that $x_1 > b$. Moreover, $P(x_o) < 1$, therefore, from (5.10) we see that $x_1 > a + \pi$.

When x_1 is substituted in the recursion equation we have

$$P(x_1) = 1 + \frac{1-4n^2}{4x_1^2} \qquad (5.11)$$

$$u_1'' + P(x_1)\, u_1 = 0$$

$$u_1(x) = k_1 \sin \sqrt{P(x_1)}\,(x-a)$$

$$x_2 = a + \pi / \sqrt{P(x_1)} \qquad (5.12)$$

We know from the inequality $x_1 > b$ that x_2 is in the interval (a,b). Trivially then $x_2 < x_1$, moreover, $P(x_1) < 1$ tells us that $x_2 > a + \pi$. Then we have

$$x_o = a < a + \pi < x_2 < b < x_1 \qquad (5.13).$$

Let us now consider the next element of the sequence. If x_2 is substituted in equations (5.2) through (5.6), we have

$$P(x_2) = 1 + \frac{1-4n^2}{4x_2^2} \qquad (5.13)$$

$$u_2'' + P(x_2)\, u_2 = 0$$

$$u_2(x) = k_2 \sin \sqrt{P(x_2)}\,(x-a)$$

$$x_3 = a + \pi/\sqrt{P(x_2)} \qquad (5.14)$$

Since x_2 is in the interval (a,b) we do not have the necessary inequality for theorem 2.1 and consequently we can no longer determine the relation of x_i to b for $i \geq 3$. We see, however, that $P(x_2) > P(x_1)$ since $P(x)$ is increasing and $x_2 < x_1$. This implies that $x_3 > x_2$. Clearly $P(x_2) < P(x_o)$ and $x_3 < x_1$. Then we have

$$x_o = a < a + \pi < x_2 < x_3 < x_1 \qquad (5.15)$$

Suppose, for some $k \geq 3$ that the following inequalities hold:

$$a < a + \pi \leq x_2 < \ldots < x_{k-2} < x_k < x_{k-1} < \ldots < x_1 \qquad (5.16).$$

The recursion formula gives

$$P(x_{k-2}) = 1 + \frac{1-4n^2}{4x_{k-2}^2} \qquad (5.17)$$

$$P(x_k) = 1 + \frac{1-4n^2}{4x_k^2} \qquad (5.18)$$

$$P(x_{k-1}) = 1 + \frac{1-4n^2}{4x_{k-1}^2} \qquad (5.19)$$

$$u''_{k-2} + P(x_{k-2})\, u_{k-2} = 0 \qquad (5.20)$$

$$u''_k + P(x_k)\, u_k = 0 \qquad (5.21)$$

$$u''_{k-1} + P(x_{k-1})\, u_{k-1} = 0 \qquad (5.22)$$

$$x_{k-1} = a + \pi / \sqrt{P(x_{k-2})}\,(x-a) \qquad (5.23)$$

$$x_k = a + \pi / \sqrt{P(x_{k-1})}\,(x-a) \qquad (5.24)$$

$$x_{k+1} = a + \pi / \sqrt{P(x_k)}\,(x-a) \qquad (5.25)$$

The assumption (5.16) tells us that $P(x_{k-2}) > P(x_k) > P(x_{k-1})$ since $P(x)$ is increasing. If we apply theorem 2.1, first with respect to (5.21) and (5.20), then with respect to (5.21) and (5.22); we have

$$x_0 = a < a + \pi < x_2 < \ldots < x_{k-2} < x_k < x_{k+1} < x_{k-1} < \ldots < x_1 \qquad (5.26)$$

We have now shown inductively that x_{i+1} lies in the interval (x_i, x_{i-1}) for all i. Let us now select the subsequence $\{z_j\}$ of $\{x_i\}$ such that for each j, $z_j = x_{2i}$. Trivially $z_1 = x_2 > z_0 = x_0$. Assuming then $z_m > z_{m-1}$ for some $m \geq 0$ we see that $z_m = x_{2m} > x_{2m-2} = z_{m-1}$. Moreover, we know that x_{2m+2} lies in the interval (x_{2m}, x_{2m+1}), therefore, we conclude $z_{m+1} > z_m$, and that $\{z_j\}$ is monotone increasing.

Clearly, the sequence $\{z_j\}$ is bounded above by x_1. Therefore, we conclude that the sequence $\{z_j\}$ converges to some $\bar{z}$.

Similarly, consider the subsequence $\{v_n\}$ of $\{x_i\}$ chosen such that for each n, $v_n = x_{2n+1}$. By an argument similar to the one for $\{z_j\}$ we can show that the subsequence $\{v_n\}$ is decreasing and bounded below by a, therefore, it converges to some $\bar{v}$. We have now shown that the sequence $\{x_i\}$ contains two convergent subsequences $\{z_j\}$ and $\{v_n\}$, and in order to prove convergence of $\{x_i\}$ we must show that the subsequences converge to the same point, i.e. that $\bar{z} = \bar{v}$. We know from (5.25) that

$$\bar{z} = a + \frac{\pi}{\sqrt{P(\bar{v})}} = a + \frac{\pi}{\sqrt{1 + \dfrac{1-4n^2}{4\left(a + \dfrac{\pi}{\sqrt{1 + \dfrac{1-4n^2}{4\bar{z}^2}}}\right)^2}}} \tag{5.26}$$

and

$$\bar{v} = a + \frac{\pi}{\sqrt{1 + \dfrac{1-4n^2}{4\left(a + \dfrac{\pi}{\sqrt{1 + \dfrac{1-4n^2}{4\bar{v}^2}}}\right)^2}}} \tag{5.27}$$

Equation (5.26), after considerable algebraic manipulation, can be reduced to the equation

$$[(\bar{z}-a)^2(4\{a(4\bar{z}^2 + 1-4n^2) + 4\bar{z}^2\pi^2\} + (1-4n^2)(4\bar{z}^2 + 1-4n^2) - 4\pi^2 \{a(4\bar{z}^2 + 1-4n^2) + 4\bar{z}^2\pi^2\}]^2 - 16a^2\bar{z}^2(4\bar{z}^2 + 1-4n^2)\{(\bar{z}-a)^2 - 4\pi^2\}^2 = 0. \tag{5.28}$$

which is clearly a polynomial of degree 8. Similarly, Equation (5.27) will reduce to the same polynomial. Then we can conclude that $\bar{z} = \bar{v}$ and the sequence $\{x_i\}$ converges to some $\hat{x}$ such that

$$\tilde{x} = a + \frac{\pi}{\sqrt{1 + \frac{1-4n^2}{4\tilde{x}^2}}}$$

which reduces to

$$4\tilde{x}^4 - 8a\tilde{x}^3 + (4a^2 + 1 - 4n^2 - 4\pi^2)\tilde{x}^2 - 2a(1-4n^2)\tilde{x}$$

$$+ (1-4n^2)\, a^2 = 0 \qquad (5.29).$$

Moreover, $\tilde{x}$ is the root of equation (5.29) nearest $a + \pi$.

Case II:

Suppose we choose $x_o = a + \pi$ then (5.3) through (5.6) becomes

$$P(x_o) = 1 + \frac{1-4n^2}{4x_o^2} = 1 + \frac{1-4n^2}{4(a+\pi)^2} \qquad (5.28)$$

$$u_o'' + P(x_o)\, u_o = 0$$

$$u_o(x) = k_o \sin\sqrt{P(x_o)}\,(x-a)$$

$$x_1 = a + \pi / \sqrt{P(x_o)} \qquad (5.29).$$

We have chosen an x_o with interval (a,b), therefore, we cannot make any comparisons between x_1 and b. Since

$P(x_o) < 1$ we know $x_1 > a + \pi = x_o$.

Let us now find the next element x_2 of the sequence $\{x_i\}$. We have the equations

$$P(x_1) = 1 + \frac{1-4n^2}{4x_1^2}$$

$$u_1'' + P(x_1)\, u_1 = 0$$

$$u_1(x) = k_1 \sin\sqrt{P(x_1)}\,(x-a)$$

$$x_2 = a + \frac{\pi}{\sqrt{P(x_1)}} .$$

The inequality $x_1 > x_o$ implies that $P(x_1) < P(x_o)$ since $P(x)$ is decreasing. Theorem 2.1, therefore, tells us that $x_2 < x_1$, moreover, from $P(x_1) < 1$ we see that $x_2 > a + \pi$ giving

$$x_o = a + \pi < x_2 < x_1 \qquad (5.30).$$

Clearly, for all $i \geq 2$ this sequence behaves exactly as the sequence of Case I, therefore, converges to some x_1' such that

$$\tilde{x}_1 = a + \frac{\pi}{\sqrt{1 + \frac{1-4n^2}{4\tilde{x}^2}}} \tag{5.31}$$

But equation (5.31) after simplification reduces to the same quartic as (5.27), and we conclude that $\tilde{x}_1 = \tilde{x}$.

In either case for $n \geq 1$ we have been unable to relate $\tilde{x}$ to b, therefore, we must consider computations for the first 20 roots.

The results of the computation displayed in table XII are comparable to those shown in table I for the case $n = 0$, as should well be expected. We see that for the 20th root, the error in table XII is 10^{-4} while for J_o, it is about 5×10^{-6}.

Suppose, however, the sequence $\{x_i\}$ were to converge to b. The final result would have, for some $\mathcal{X}$, the following

$$b = a + \frac{\pi}{\sqrt{P(\mathcal{X})}}$$

$$= a + \frac{\pi}{\sqrt{1 + \frac{1-4n^2}{4\mathcal{X}^2}}}$$

$$(4\mathcal{X}^2 + 1 - 4n^2)(b-a)^2 - 4\mathcal{X}^2\pi^2 = 0$$

which when solved for $\mathcal{X}$ shows that

$$\mathcal{X} = \frac{(b-a)}{4} \sqrt{\frac{4n^2-1}{((b-a)^2 - \pi^2)}} \tag{5.32}$$

Trivially $\mathcal{X}$ determined in this manner is real valued, since $(b-a) > \pi$ and $4n^2 > 1$. The values of $\mathcal{X}$ computed from equation (5.32) are shown in table XIII. These values suggest that the following modification be made to the recursion formulae.

$$P(\bar{x}_i) = 1 + \frac{1-4n^2}{4\left(\frac{a+\bar{x}_i}{2}\right)^2}$$

$$u_i'' + P(\bar{x}_i)u_i = 0$$

$$u_i(x) = k_i \sin \sqrt{P(\bar{x}_i)}(x-a)$$

$$\bar{x}_{i+1} = a + \frac{\pi}{\sqrt{P(\bar{x}_i)}}$$

We have already shown that the sequence $\{x_i\}$ defined by equations (5.3), (5.4), (5.5) and (5.6) converges to $\tilde{x}$. The sequence $\{\bar{x}_i\}$ is the sum of a constant and a multiple of x_i, and therefore will converge. This implies that the sequence $\{\bar{x}_i\}$ converges to some $\bar{x}$ such that

$$\bar{x} = a + \frac{\pi}{\sqrt{1 + \frac{1-n^2}{4\left(\frac{a+\bar{x}}{2}\right)^2}}} \qquad (5.13)$$

$$\bar{x}^4 + (1-4n^2-2a^2-\pi^2)\bar{x}^2 - [2a(1-4n^2+\pi^2)]\bar{x} + a^4 + a^2(1-4n^2-\pi^2) = 0 \qquad (5.14)$$

The roots of the quartic (5.14) nearest $a+\pi$ are displayed in table XIV for the first 20 roots of $J_1(x)$. Comparing the errors of table XIV against table XII we see that $|\bar{x}-b|$ is one order of magnitude smaller than $|\tilde{x}-b|$.

We shall next examine the sequence $\{\hat{x}_i\}$, i.e. the mean value of $P(x)$ over the interval (a, x). The mean value of $P(x)$ is defined in the same manner as in section 4.2. We know that the sequence $\{\hat{x}_i\}$ will converge since it consists of elements which are multiples of the elements of a convergent sequence. Then we have

$$\hat{x} = a + \pi / \sqrt{P(\hat{x})}$$

$$= a + \frac{\pi}{\sqrt{1 + \frac{1-4n^2}{4a\hat{x}}}}$$

$$4a\hat{x}^3+(1-4n^2-8a^2)\hat{x}^2+(4a^3-2a-8a^2n^2-4a\pi)\hat{x}\; a^3+4a^4n^2=0 \qquad (5.15)$$

We see that for $n \geq 1$, the limit of the sequence results in a cubic similar to the one for $\hat{x}$ when $n = 0$. Again we know that $\hat{x}$ is the root of the cubic nearest $a + \pi$.

Table XV displays the roots of equation (5.15) for the first 20 roots of $J_1(x)$. As expected from the results for $J_o(x)$, we see that again $\hat{x}$ is past b and that $\bar{x}$ and $\hat{x}$ have straddled b. We note, as for $J_o(x)$, $|\hat{x}-b| > |\bar{x}-b|$.

We shall now examine $\overline{x1}$ and $\overline{x2}$, as defined in section 4.2. The values of $\overline{x1}$ are displayed in table XVI. Again, as for $J_o(x)$, we see that $\overline{x1}$ is further from b than $\bar{x}$.

$\overline{x2}$ is again nearest b for $J_1(x)$, with a maximum error for the first root. In general, we see that the various sequences generated for $J_1(x)$ tend to converge to points farther from b than the same sequence for $J_o(x)$.

As expected, the cumulative errors shown in tables XVIII through XXII exhibit the same characteristics as the cumulative errors for $J_o(x)$.

6. Conclusions.

We have proposed a technique of computing the consecutive zeros of a solution to a Sturm-Liouville System.

$$[r(x)y'(x)]' + P(x)y(x) = 0 \tag{6.1}$$

$$y(a) = 0 \qquad y(b) = 0$$

The technique requires the determination of an $\tilde{x}$ such that the equation

$$[r(\tilde{x})u'(x)]' + P(\tilde{x})y(x) = 0 \tag{6.2}$$

will have a solution $u(x)$ which may be forced to vanish at $x = a$ and at $x = b$.

The existence of such an $\tilde{x}$ has been shown by theorem 2.2. Moreover, in theorem 2.2.1 we have established upper and lower bounds on the range of this $\tilde{x}$.

In the specific case of the Bessel functions, we note that each of the sequences generated by the proposed technique converges to a particular x, no matter whether the first element was chosen equal to the known root a, or chosen equal to $a + \pi$. We see that $\tilde{x}$ turns out to be the root of a quartic nearest $a + \pi$. Similarly, we see that $\bar{x}$ is also a root of a quartic polynomial while $\hat{x}$ resolves into a cubic equation.

The solution for $\mathcal{X}$ for the case $n = 0$ and for $n = 1$ both indicate that the value of the desired x as defined by theorem 2.2 is almost midway between a and b. The difference between $\tilde{x}$ and $\mathcal{X}$ approaches $\pi/2$ as x increases.

We have also shown that the desired x for the Bessel functions lies between the x determined by a linear averaging technique and the x determined by the mean value of $P(x)$ over the interval (a, x).

Unfortunately, all of the above generating techniques failed to result in a sequence converging to the desired x.

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TABLE I

VALUES OF $\tilde{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(X(I)) = 1+1/4(X(I))^2$ $\qquad$ $X(I+1) = A+\pi/\sqrt{P(X(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5336719233	.0135938131
2	8.6537279126	8.6564432415	.0027153292
3	11.7915344390	11.7925004759	.0009660370
4	14.9309177082	14.9313671612	.0004494532
5	18.0710639674	18.0713085639	.0002445965
6	21.2116366294	21.2117842031	.0001475737
7	24.3524715304	24.3525673207	.0000957905
8	27.4934791317	27.4935447974	.0000656658
9	30.6346064676	30.6346534281	.0000469606
10	33.7758202134	33.7758549489	.0000347356
11	36.9170983527	36.9171247659	.0000264135
12	40.0584257636	40.0584463133	.0000205500
13	43.1997917127	43.1998080136	.0000163012
14	46.3411883712	46.3412015187	.0000131483
15	49.4826098960	49.4826206546	.0000107595
16	52.6240518400	52.6240607537	.0000089143
17	55.7655107528	55.7655182229	.0000074702
18	58.9069839241	58.9069902431	.0000063197
19	62.0484691886	62.0484745828	.0000053946
20	65.1899647992	65.1899694391	.0000046415

TABLE II

VALUES OF χ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

CHI IS THE VALUE X MUST ASSUME TO CONVERGE TO B

$\chi = ((B-A)/2)(\sqrt{1/((B-A)^2-\pi^2)})$

ERROR IS THE DIFFERENCE BETWEEN X AND CHI

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	3.8368678503	1.6968040731
2	8.6537279126	7.0180586502	1.6383845913
3	11.7915344390	10.1751194785	1.6173809974
4	14.9309177082	13.3249362609	1.6064309003
5	18.0710639674	16.4716300704	1.5996784936
6	21.2116366294	19.6166964089	1.5950877941
7	24.3524715304	22.7608070695	1.5917602512
8	27.4934791317	25.9043037482	1.5892410493
9	30.6346064676	29.0473744823	1.5872789458
10	33.7758202134	32.1902150838	1.5856398651
11	36.9170983527	35.3327245601	1.5844002059
12	40.0584257636	38.4751889762	1.5832573371
13	43.1997917127	41.6175294435	1.5822785702
14	46.3411883712	44.7596824104	1.5815191083
15	49.4826098960	47.9016739726	1.5809466820
16	52.6240518400	51.0439078268	1.5801529270
17	55.7655107528	54.1856175233	1.5799006997
18	58.9069839241	57.3278564233	1.5791338198
19	62.0484691886	60.4698329102	1.5786416726
20	65.1899647992	63.6115127150	1.5784567241

TABLE III

VALUES OF $\bar{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\bar{X}(I)) = 1+1/4(((A+\bar{X}(I))/2))^2 \qquad \bar{X}(I+1) = A+\pi/\sqrt{P(\bar{X}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5217120022	.0016338920
2	8.6537279126	8.6538810786	.0001531661
3	11.7915344390	11.7915695005	.0000350617
4	14.9309177082	14.9309296827	.0000119747
5	18.0710639674	18.0710691069	.0000051396
6	21.2116366294	21.2116391868	.0000025579
7	24.3524715304	24.3524729425	.0000014123
8	27.4934791317	27.4934799736	.0000008423
9	30.6346064676	30.6346070007	.0000005335
10	33.7758202134	33.7758205654	.0000003527
11	36.9170983527	36.9170985967	.0000002443
12	40.0584257636	40.0584259368	.0000001732
13	43.1997917127	43.1997918384	.0000001260
14	46.3411883712	46.3411884652	.0000000946
15	49.4826098960	49.4826099686	.0000000730
16	52.6240518400	52.6240518950	.0000000556
17	55.7655107528	55.7655107975	.0000000452
18	58.9069839241	58.9069839586	.0000000349
19	62.0484691886	62.0484692166	.0000000280
20	65.1899647992	65.1899648216	.0000000226

TABLE IV

VALUES OF $\hat{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\hat{X}(I)) = 1+1/4A\hat{X}(I)$ $\qquad \hat{X}(I+1) = A+\tau/\sqrt{P(\hat{X}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5172325156	-.0028455946
2	8.6537279126	8.6534819095	-.0002460030
3	11.7915344390	11.7914791510	-.0000552879
4	14.9309177082	14.9308989581	-.0000187501
5	18.0710639674	18.0710559469	-.0000080201
6	21.2116366294	21.2116326448	-.0000039842
7	24.3524715304	24.3524693325	-.0000021976
8	27.4934791317	27.4934778223	-.0000013094
9	30.6346064676	30.6346056401	-.0000008275
10	33.7758202134	33.7758196630	-.0000005496
11	36.9170983527	36.9170979746	-.0000003774
12	40.0584257636	40.0584254945	-.0000002689
13	43.1997917127	43.1997915152	-.0000001969
14	46.3411883712	46.3411882240	-.0000001468
15	49.4826098960	49.4826097842	-.0000001110
16	52.6240518400	52.6240517525	-.0000000871
17	55.7655107528	55.7655106848	-.0000000672
18	58.9069839241	58.9069838692	-.0000000548
19	62.0484691886	62.0484691439	-.0000000444
20	65.1899647992	65.1899647620	-.0000000366

TABLE V

VALUES OF $\overline{X1}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\overline{X1}(I)) = (P(X(I))+P(\hat{X}(I)))/2 \qquad \overline{X1}(I+1) = A+\pi/\sqrt{P(\overline{X1}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5254224626	.0053443524
2	8.6537279126	8.6549613329	.0012334206
3	11.7915344390	11.7919896499	.0004552110
4	14.9309177082	14.9311330232	.0002153153
5	18.0710639674	18.0711822445	.0001182773
6	21.2116366294	21.2117084200	.0000717907
7	24.3524715304	24.3525183247	.0000467948
8	27.4934791317	27.4935113089	.0000321774
9	30.6346064676	30.6346295336	.0000230662
10	33.7758202134	33.7758373059	.0000170928
11	36.9170983527	36.9171113698	.0000130179
12	40.0584257636	40.0584359039	.0000101405
13	43.1997917127	43.1997997640	.0000080521
14	46.3411883712	46.3411948718	.0000065007
15	49.4826098960	49.4826152194	.0000053242
16	52.6240518400	52.6240562536	.0000044135
17	55.7655107528	55.7655144539	.0000037015
18	58.9069839241	58.9069870561	.0000031325
19	62.0484691886	62.0484718634	.0000026751
20	65.1899647992	65.1899671014	.0000023024

TABLE VI

VALUES OF $\overline{X2}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\overline{X2}(I)) = (P(\overline{X}(I)) + P(\hat{X}(I)))/2 \qquad \overline{X2}(I+1) = A + \pi/\sqrt{P(\overline{X2}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5194687883	-.0006093219
2	8.6537279126	8.6536814596	-.0000464527
3	11.7915344390	11.7915243239	-.0000101150
4	14.9309177082	14.9309143201	-.0000033879
5	18.0710639674	18.0710625267	-.0000014403
6	21.2116366294	21.2116359160	-.0000007132
7	24.3524715304	24.3524711374	-.0000003927
8	27.4934791317	27.4934788980	-.0000002336
9	30.6346064676	30.6346063204	-.0000001470
10	33.7758202134	33.7758201146	-.0000000984
11	36.9170983527	36.9170982856	-.0000000665
12	40.0584257636	40.0584257152	-.0000000478
13	43.1997917127	43.1997916764	-.0000000355
14	46.3411883712	46.3411883442	-.0000000261
15	49.4826098960	49.4826098764	-.0000000190
16	52.6240518400	52.6240518242	-.0000000158
17	55.7655107528	55.7655107416	-.0000000110
18	58.9069839241	58.9069839139	-.0000000099
19	62.0484691886	62.0484691802	-.0000000082
20	65.1899647992	65.1899647918	-.0000000070

TABLE VII

CUMULATIVE ERRORS FOR $\tilde{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\tilde{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5336719233	.0135938131
2	8.6537279126	8.6700534131	.0163255008
3	11.7915344390	11.8088337616	.0172993229
4	14.9309177082	14.9486705526	.0177528444
5	18.0710639674	18.0890637650	.0179997976
6	21.2116366294	21.2297854787	.0181488496
7	24.3524715304	24.3707171553	.0182456251
8	27.4934791317	27.5117911105	.0183119791
9	30.6346064676	30.6529659065	.0183594390
10	33.7758202134	33.7942147609	.0183945483
11	36.9170983527	36.9355196003	.0184212479
12	40.0584257636	40.0768677853	.0184420225
13	43.1997917127	43.2182502151	.0184585024
14	46.3411883712	46.3596601672	.0184717962
15	49.4826098960	49.5010925708	.0184826752
16	52.6240518400	52.6425435282	.0184916886
17	55.7655107528	55.7840099940	.0184992421
18	58.9069839241	58.9254895560	.0185056320
19	62.0484691886	62.0669802753	.0185110874
20	65.1899647992	65.2084805798	.0185157806

TABLE VIII

CUMULATIVE ERRORS FOR $\overline{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\overline{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5217120022	.0016338920
2	8.6537279126	8.6555185516	.0017906391
3	11.7915344390	11.7933614515	.0018270126
4	14.9309177082	14.9327572957	.0018395875
5	18.0710639674	18.0729090152	.0018450481
6	21.2116366294	21.2134844260	.0018477967
7	24.3524715304	24.3543208619	.0018493315
8	27.4934791317	27.4953293884	.0018502570
9	30.6346064676	30.6364573166	.0018508493
10	33.7758202134	33.7776714582	.0018512453
11	36.9170983527	36.9189498741	.0018515220
12	40.0584257636	40.0602774834	.0018517202
13	43.1997917127	43.2016435778	.0018518660
14	46.3411883712	46.3430403462	.0018519759
15	49.4826098960	49.4844619567	.0018520612
16	52.6240518400	52.6259039668	.0018521272
17	55.7655107528	55.7673629336	.0018521811
18	58.9069839241	58.9088361468	.0018522234
19	62.0484691886	62.0503214458	.0018522573
20	65.1899647992	65.1918170843	.0018522854

TABLE IX

CUMULATIVE ERRORS FOR $\hat{X}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\hat{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5172325156	-.0028455946
2	8.6537279126	8.6506294168	-.0030984957
3	11.7915344390	11.7883782729	-.0031561660
4	14.9309177082	14.9277417252	-.0031759829
5	18.0710639674	18.0678793995	-.0031845679
6	21.2116366294	21.2084477427	-.0031888865
7	24.3524715304	24.3492802321	-.0031912980
8	27.4934791317	27.4902863791	-.0031927526
9	30.6346064676	30.6314127846	-.0031936828
10	33.7758202134	33.7726259045	-.0031943080
11	36.9170983527	36.9139036089	-.0031947432
12	40.0584257636	40.0552307060	-.0031950568
13	43.1997917127	43.1965964232	-.0031952894
14	46.3411883712	46.3379929066	-.0031954644
15	49.4826098960	49.4794142973	-.0031955985
16	52.6240518400	52.6208561352	-.0031957048
17	55.7655107528	55.7623149641	-.0031957879
18	58.9069839241	58.9037880665	-.0031958568
19	62.0484691886	62.0452732751	-.0031959134
20	65.1899647992	65.1867688391	-.0031959599

TABLE X

CUMULATIVE ERRORS FOR $\overline{X1}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\overline{X1}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5254224626	.0053443524
2	8.6537279126	8.6603153765	.0065874640
3	11.7915344390	11.7985812158	.0070467769
4	14.9309177082	14.9381818194	.0072641112
5	18.0710639674	18.0784474835	.0073835164
6	21.2116366294	21.2190926266	.0074559972
7	24.3524715304	24.3599747745	.0075032441
8	27.4934791317	27.5010148650	.0075357334
9	30.6346064676	30.6421654909	.0075590234
10	33.7758202134	33.7833964955	.0075762823
11	36.9170983527	36.9246877786	.0075894268
12	40.0584257636	40.0660254285	.0075996651
13	43.1997917127	43.2073995080	.0076077954
14	46.3411883712	46.3488027304	.0076143594
15	49.4826098960	49.4902296308	.0076197354
16	52.6240518400	52.6316760313	.0076241914
17	55.7655107528	55.7731386814	.0076279288
18	58.9069839241	58.9146150155	.0076310916
19	62.0484691886	62.0561029809	.0076337926
20	65.1899647992	65.1976009160	.0076361171

TABLE XI

CUMULATIVE ERRORS FOR $\overline{X2}$ *** $J_0(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ZERO FOR ROOTS ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\overline{X2}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	5.5200781103	5.5194687883	-.0006093219
2	8.6537279126	8.6530707316	-.0006571810
3	11.7915344390	11.7908666495	-.0006677894
4	14.9309177082	14.9302463082	-.0006714000
5	18.0710639674	18.0703910086	-.0006729586
6	21.2116366294	21.2109628869	-.0006737421
7	24.3524715304	24.3517973502	-.0006741801
8	27.4934791317	27.4928046870	-.0006744443
9	30.6346064676	30.6339318538	-.0006746134
10	33.7758202134	33.7751454851	-.0006747281
11	36.9170983527	36.9164235452	-.0006748069
12	40.0584257636	40.0577508984	-.0006748646
13	43.1997917127	43.1991168046	-.0006749080
14	46.3411883712	46.3405134305	-.0006749401
15	49.4826098960	49.4819349311	-.0006749644
16	52.6240518400	52.6233768547	-.0006749846
17	55.7655107528	55.7648357525	-.0006749997
18	58.9069839241	58.9063089108	-.0006750130
19	62.0484691886	62.0477941642	-.0006750239
20	65.1899647992	65.1892897654	-.0006750335

TABLE XII

VALUES OF $\tilde{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(X(I)) = 1+(1-4N^2)/4(X(I))^2 \qquad X(I+1) = A+\pi/\sqrt{P(X(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	6.9976376109	-.0179490588
2	10.1734681348	10.1686351451	-.0048329895
3	13.3236919360	13.3217202593	-.0019716767
4	16.4706300506	16.4696368496	-.0009932009
5	19.6158585101	19.6152890925	-.0005694173
6	22.7600853802	22.7597279320	-.0003574477
7	25.9036720875	25.9034342724	-.0002378150
8	29.0468285335	29.0466620042	-.0001665289
9	32.1896799095	32.1895587798	-.0001211293
10	35.3323075492	35.3322166996	-.0000908489
11	38.4747662339	38.4746963540	-.0000698795
12	41.6170942122	41.6170393117	-.0000549003
13	44.7593189972	44.7592750816	-.0000439151
14	47.9014608869	47.9014252098	-.0000356767
15	51.0435351832	51.0435058065	-.0000293767
16	54.1855536401	54.1855291631	-.0000244770
17	57.3275254359	57.3275048267	-.0000206090
18	60.4694578443	60.4694403261	-.0000175173
19	63.6113566970	63.6113416851	-.0000150119
20	66.7532267310	66.7532137688	-.0000129620

TABLE XIII

VALUES OF χ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

CHI IS THE VALUE X MUST ASSUME TO CONVERGE TO B

$$\chi = ((B-A)/2)(\sqrt{(1-4N^2)/((B-A)^2-\pi^2)})$$

ERROR IS THE DIFFERENCE BETWEEN X AND CHI

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	5.3312873839	1.6663502270
2	10.1734681348	8.5374782400	1.6311569051
3	13.3236919360	11.7071112888	1.6146089705
4	16.4706300506	14.8645464401	1.6050904095
5	19.6158585101	18.0163550223	1.5989340702
6	22.7600853802	21.1610798943	1.5986480378
7	25.9036720875	24.3119656080	1.5914686644
8	29.0468285335	27.4576333771	1.5890286271
9	32.1896799095	30.6024408266	1.5871179532
10	35.3323075492	33.7466444736	1.5855722260
11	38.4747662339	36.8904200587	1.5842762953
12	41.6170942122	40.0338375270	1.5832017846
13	44.7593189972	43.1769980257	1.5822770558
14	47.9014608869	46.3199358247	1.5814893851
15	51.0435351832	49.4627229534	1.5807828531
16	54.1855536401	52.6053818800	1.5801472832
17	57.3275254359	55.7479370888	1.5795677379
18	60.4694578443	58.8902029432	1.5792373829
19	63.6113566970	62.0326487795	1.5786929056
20	66.7532267310	65.1751772780	1.5780364908

TABLE XIV

VALUES OF $\overline{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE $P(\overline{X}(I)) = 1+(1-4N^2)/4((A+\overline{X}(I))/2)^2$ $\overline{X}(I+1) = A+\pi/\sqrt{P(\overline{X}(I))}$ ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0141418219	-.0014448478
2	10.1734681348	10.1732513588	-.0002167758
3	13.3236919360	13.3236309085	-.0000610273
4	16.4706300506	16.4706066190	-.0000234314
5	19.6158585101	19.6158476649	-.0000108451
6	22.7600853802	22.7600786900	-.0000066901
7	25.9036720875	25.9036688204	-.0000032669
8	29.0468285335	29.0468265270	-.0000020061
9	32.1896799095	32.1896786084	-.0000013006
10	35.3323075492	35.3323066691	-.0000008800
11	38.4747662339	38.4747656174	-.0000006158
12	41.6170942122	41.6170937680	-.0000004442
13	44.7593189972	44.7593186684	-.0000003282
14	47.9014608869	47.9014606383	-.0000002479
15	51.0435351832	51.0435349923	-.0000001903
16	54.1855536401	54.1855534911	-.0000001483
17	57.3275254359	57.3275253186	-.0000001170
18	60.4694578443	60.4694577483	-.0000000957
19	63.6113566970	63.6113566197	-.0000000766
20	66.7532267310	66.7532266676	-.0000000615

TABLE XV

VALUES OF $\hat{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\hat{X}(I)) = 1 + (1 - 4N^2)/\ 4A\hat{X}(I) \qquad \hat{X}(I+1) = A + \pi/\sqrt{P(\hat{X}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0180468041	.0024601344
2	10.1734681348	10.1738162260	.0003480912
3	13.3236919360	13.3237883153	.0000963795
4	16.4706300506	16.4706667955	.0000367450
5	19.6158585101	19.6158754556	.0000169458
6	22.7600853802	22.7600932522	.0000078722
7	25.9036720875	25.9036771739	.0000050868
8	29.0468285335	29.0468316572	.0000031241
9	32.1896799095	32.1896819314	.0000020222
10	35.3323075492	35.3323089145	.0000013660
11	38.4747662339	38.4747671895	.0000009564
12	41.6170942122	41.6170949005	.0000006892
13	44.7593189972	44.7593195057	.0000005093
14	47.9014608869	47.9014612706	.0000003842
15	51.0435351832	51.0435354784	.0000002958
16	54.1855536401	54.1855538711	.0000002316
17	57.3275254359	57.3275256194	.0000001842
18	60.4694578443	60.4694579896	.0000001462
19	63.6113566970	63.6113568163	.0000001198
20	66.7532267310	66.7532268297	.0000000997

TABLE XVI

VALUES OF $\overline{X1}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\overline{X1}(I)) = (P(X(I))+P(\hat{X}(I)))/2 \qquad \overline{X1}(I) = A+\pi/\sqrt{P(\overline{X1}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0077901660	-.0077965038
2	10.1734681348	10.1712216930	-.0022464416
3	13.3236919360	13.3227535994	-.0009383365
4	16.4706300506	16.4701516442	-.0004784064
5	19.6158585101	19.6155822147	-.0002762953
6	22.7600853802	22.7599105686	-.0001748114
7	25.9036720875	25.9035557127	-.0001163747
8	29.0468285335	29.0467468258	-.0000817076
9	32.1896799095	32.1896203524	-.0000595564
10	35.3323075492	35.3322628057	-.0000447430
11	38.4747662339	38.4747317713	-.0000344625
12	41.6170942122	41.6170671051	-.0000271062
13	44.7593189972	44.7592972936	-.0000217033
14	47.9014608869	47.9014432402	-.0000176465
15	51.0435351832	51.0435206424	-.0000145406
16	54.1855536401	54.1855415171	-.0000121228
17	57.3275254359	57.3275152231	-.0000102125
18	60.4694578443	60.4694491578	-.0000086856
19	63.6113566970	63.6113492502	-.0000074461
20	66.7532267310	66.7532202993	-.0000064312

TABLE XVII

VALUES OF $\overline{X2}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

$P(\overline{X2}(I)) = (P(\bar{X}(I))+P(\hat{X}(I)))/2 \qquad \overline{X2}(I) = A+\pi/\sqrt{P(\overline{X2}(I))}$

ERROR IS THE DIFFERENCE BETWEEN TABULATED AND COMPUTED ROOTS

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0160913781	.0005047083
2	10.1734681348	10.1735337211	.0000655864
3	13.3236919360	13.3237096060	.0000176702
4	16.4706300506	16.4706367063	.0000066559
5	19.6158585101	19.6158615602	.0000030501
6	22.7600853802	22.7600859711	.0000005910
7	25.9036720875	25.9036729969	.0000009099
8	29.0468285335	29.0468290923	.0000005590
9	32.1896799095	32.1896802699	.0000003608
10	35.3323075492	35.3323077913	.0000002430
11	38.4747662339	38.4747664034	.0000001703
12	41.6170942122	41.6170943342	.0000001225
13	44.7593189972	44.7593190875	.0000000905
14	47.9014608869	47.9014609549	.0000000681
15	51.0435351832	51.0435352353	.0000000527
16	54.1855536401	54.1855536811	.0000000416
17	57.3275254359	57.3275254695	.0000000336
18	60.4694578443	60.4694578694	.0000000253
19	63.6113566970	63.6113567185	.0000000216
20	66.7532267310	66.7532267496	.0000000191

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TABLE XVIII

CUMULATIVE ERRORS FOR $\hat{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\hat{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST RCOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	6.9976376109	-.0179490588
2	10.1734681348	10.1507267661	-.0227413687
3	13.3236919360	13.2990017352	-.0246902007
4	16.4706300506	16.4449597476	-.0256703027
5	19.6158585101	19.5896268408	-.0262316693
6	22.7600853802	22.7335015247	-.0265838551
7	25.9036720875	25.8768550311	-.0268170560
8	29.0468285335	29.0198475332	-.0269810002
9	32.1896799095	32.1625796901	-.0271002193
10	35.3323075492	35.3051179312	-.0271896177
11	38.4747662339	38.4475078629	-.0272583706
12	41.6170942122	41.5897818329	-.0273123789
13	44.7593189972	44.7319634212	-.0273555757
14	47.9014608869	47.8740702206	-.0273906654
15	51.0435351832	51.0161156254	-.0274195571
16	54.1855536401	54.1581100114	-.0274436282
17	57.3275254359	57.3000615416	-.0274638942
18	60.4694578443	60.4419767251	-.0274811186
19	63.6113566970	63.5838608174	-.0274958793
20	66.7532267310	66.7257181071	-.0275086237

TABLE XIX

CUMULATIVE ERRORS FOR $\bar{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\bar{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0141418219	-.0014448478
2	10.1734681348	10.1718119476	-.0016561872
3	13.3236919360	13.3219771462	-.0017147896
4	16.4706300506	16.4688930572	-.0017369931
5	19.6158585101	19.6141113704	-.0017471395
6	22.7600853802	22.7583319838	-.0017533960
7	25.9036720875	25.9019167111	-.0017553763
8	29.0468285335	29.0450713504	-.0017571828
9	32.1896799095	32.1879215697	-.0017583393
10	35.3323075492	35.3305484364	-.0017591120
11	38.4747662339	38.4730065875	-.0017596461
12	41.6170942122	41.6153341858	-.0017600260
13	44.7593189972	44.7575586932	-.0017603031
14	47.9014608869	47.8997003762	-.0017605102
15	51.0435351832	51.0417745160	-.0017606668
16	54.1855536401	54.1837928528	-.0017607870
17	57.3275254359	57.3257645555	-.0017608804
18	60.4694578443	60.4676968884	-.0017609558
19	63.6113566970	63.6095956815	-.0017610151
20	66.7532267310	66.7514656689	-.0017610620

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TABLE XX

CUMULATIVE ERRORS FOR $\hat{X}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\hat{X}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0180468041	.0024601344
2	10.1734681348	10.1762664455	.0027983109
3	13.3236919360	13.3265823792	.0028904433
4	16.4706300506	16.4735551225	.0029250719
5	19.6158585101	19.6187993325	.0029408227
6	22.7600853802	22.7630333365	.0029479564
7	25.9036720875	25.9066256429	.0029535557
8	29.0468285335	29.0497848745	.0029563413
9	32.1896799095	32.1926380284	.0029581190
10	35.3323075492	35.3352668518	.0029593028
11	38.4747662339	38.4777263533	.0029601197
12	41.6170942122	41.6200549109	.0029606996
13	44.7593189972	44.7622801177	.0029611211
14	47.9014608869	47.9044223214	.0029614345
15	51.0435351832	51.0464968551	.0029616724
16	54.1855536401	54.1885154955	.0029618555
17	57.3275254359	57.3304874348	.0029619992
18	60.4694578443	60.4724199548	.0029621108
19	63.6113566970	63.6143188979	.0029622011
20	66.7532267310	66.7561890054	.0029622754

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TABLE XXI

CUMULATIVE ERRORS FOR $\overline{X1}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\overline{X1}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0077901660	-.0077965038
2	10.1734681348	10.1634497307	-.0100184040
3	13.3236919360	13.3127478317	-.0109441041
4	16.4706300506	16.4592144503	-.0114155999
5	19.6158585101	19.6041707369	-.0116877730
6	22.7600853802	22.7482254347	-.0118599452
7	25.9036720875	25.8916985528	-.0119735343
8	29.0468285335	29.0347745535	-.0120539797
9	32.1896799095	32.1775672976	-.0121126117
10	35.3323075492	35.3201508904	-.0121566580
11	38.4747662339	38.4625756508	-.0121905831
12	41.6170942122	41.6048769467	-.0122172653
13	44.7593189972	44.7470803680	-.0122386289
14	47.9014608869	47.8892048877	-.0122559989
15	51.0435351832	51.0312648714	-.0122703116
16	54.1855536401	54.1732713953	-.0122822442
17	57.3275254359	57.3152331384	-.0122922968
18	60.4694578443	60.4571569972	-.0123008466
19	63.6113566970	63.5990485204	-.0123081761
20	66.7532267310	66.7409122232	-.0123145066

TABLE XXII

CUMULATIVE ERRORS FOR $\overline{X2}$ *** $J_1(X)$

ZEROS OF THE BESSEL FUNCTION OF ORDER ONE FOR ROOTS NUMBER ONE THROUGH TWENTY, BASED ON THE RECURSION FORMULAE

AFTER THE FIRST ROOT IS COMPUTED, THE NEXT ROOT IS COMPUTED FROM THE VALUE OF $\overline{X2}$ LAST OBTAINED. ERROR IS CUMULATIVE FROM THE FIRST ROOT.

ROOT NO.	TABULATED ROOT	COMPUTED ROOT	ERROR
1	7.0155866698	7.0160913781	.0005047083
2	10.1734681348	10.1740364628	.0005683281
3	13.3236919360	13.3242770866	.0005851508
4	16.4706300506	16.4712214330	.0005913828
5	19.6158585101	19.6164527028	.0005941927
6	22.7600853802	22.7606800152	.0005946353
7	25.9036720875	25.9042685339	.0005964467
8	29.0468285335	29.0474254708	.0005969373
9	32.1896799095	32.1902771583	.0005972489
10	35.3323075492	35.3329050038	.0005974551
11	38.4747662339	38.4753638301	.0005975968
12	41.6170942122	41.6176919080	.0005976967
13	44.7593189972	44.7599167656	.0005977688
14	47.9014608869	47.9020587085	.0005978224
15	51.0435351832	51.0441330457	.0005978626
16	54.1855536401	54.1861515343	.0005978945
17	57.3275254359	57.3281233553	.0005979196
18	60.4694578443	60.4700557813	.0005979377
19	63.6113566970	63.6119546490	.0005979527
20	66.7532267310	66.7538246959	.0005979659

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